

2-TO-1 MAPS ON HEREDITARILY INDECOMPOSABLE CONTINUA

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ABSTRACT. Suppose f is a 2-to-1 continuous map from the hereditarily indecomposable continuum X onto a continuum Y . In order for it to be the case that each proper subcontinuum C in Y has as its preimage two disjoint continua each of which f maps homeomorphically onto C , it is obviously necessary that f satisfy the condition that each nondense connected subset of Y has disconnected preimage. We show that this condition is also sufficient, and thus any 2-to-1 continuous map from a hereditarily indecomposable continuum satisfying this condition must be confluent and have an image that is hereditarily indecomposable.

I. INTRODUCTION

It is not known if there is a continuous 2-to-1 map defined on a pseudoarc. (A function is (exactly) 2-to-1 if each image point has exactly two preimage points.) This question, first asked by Mioduszewski in the 1960s, is a special case of the more general question of whether any arclike indecomposable continuum can support a 2-to-1 map. The arc itself does not admit even a finitely discontinuous 2-to-1 function [3]. An example of an arclike (but decomposable) continuum that supports a 2-to-1 map was constructed by Wayne Lewis and is described in [2]. Also, Ira Rosenholtz announced such a continuum in [8]. This paper studies 2-to-1 maps defined on hereditarily indecomposable continua in general, but with the pseudoarc in mind.

It was shown by H. Cook [1] that if the image Y of a map f defined on a continuum is hereditarily indecomposable then f is confluent; therefore if the map is also 2-to-1 then each continuum Z in Y either has as its inverse under f two disjoint continua each of which maps homeomorphically onto Z , or $f^{-1}(Z)$ is a single continuum that maps 2-to-1 onto Z . If we first restrict f so that the new image is a subcontinuum Y' of Y minimal with respect to having a connected preimage, then only the first case occurs, i.e. for the restricted map, each proper subcontinuum Z in Y' has a preimage consisting of two disjoint continua and each maps 1-to-1 onto Z . The author does not know if such a map can be defined on the pseudoarc but the structure of such

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a map is quite clear. Note that McLean [6] has shown that the confluent image of the pseudoarc must be treelike; despite all this the following is apparently unknown:

Question. *Suppose f is a 2-to-1 continuous function from the pseudoarc onto a hereditarily indecomposable continuum Y . Must Y be the pseudoarc?*

This paper concentrates on 2-to-1 continuous maps defined on hereditarily indecomposable continua where the image Y is not known to be hereditarily indecomposable. With no assumptions on Y , it is shown that if f satisfies Assumption 1 (i.e. if C is a connected nondense subset of the image then $f^{-1}(C)$ is not connected) then f is confluent; this means that f has the same clear structure as that guaranteed for minimal 2-to-1 functions with hereditarily indecomposable images. (See Theorem 1 in §III.) It also follows that if X is hereditarily indecomposable and f satisfies Assumption 1, then in fact Y must be hereditarily indecomposable.

All spaces in this paper are assumed metric. There is a glossary in the back for common terms from continuum theory, and specialized definitions are in §II; §III contains the main result of this paper and its proof.

II. DEFINITIONS AND REFERENCE THEOREMS

There is an example at the end of this section illustrating these reference facts and definitions. (See Illustration 1.)

Definition. For each point p in the domain A of a 2-to-1 map f , let p^\wedge , called the *twin* of p , denote the other point of A that maps the same as p under f . The *twin function* from A onto itself takes each x to x^\wedge . For any subset D of A , define its twin to be $D^\wedge = \{x^\wedge : x \text{ is in } D\}$.

Definition. The point p in the domain A of a 2-to-1 map is a *hinge point* if there is a sequence of points in A converging to p whose twin sequence also converges to p . If D is a subset of A , then p is a *hinge point for points of D* if there is a pair of twin sequences from D converging to p .

Definition. A function $t : A \rightarrow A$ is *semicontinuous* (Mioduszewski, [7, p. 12]) if for any point x in A and any sequence of points converging to x , the sequence of images under t does not converge to any point other than $t(x)$ or x .

Fact 0 (Mioduszewski, [7, p. 12]). *If $f : A \rightarrow B$ is 2-to-1 and continuous, then the twin function on A is semicontinuous. Note that if A is compact then the set of points at which the twin function is not continuous is exactly the set of hinge points.*

Reference (Interior Lemma) [3]. *Suppose $f : A \rightarrow B$ is continuous and 2-to-1, and A' is a compact subset of A such that $f(A') = B$. Then A' must have interior.*

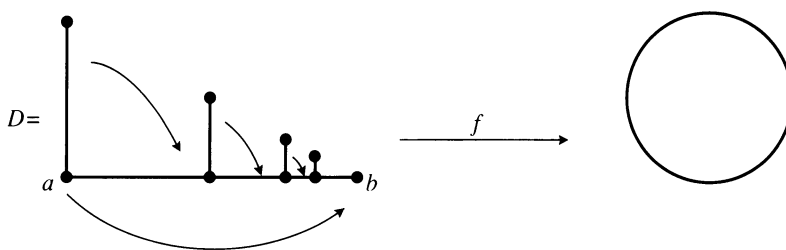


ILLUSTRATION 1

Reference Theorem 2 [3]. Suppose $f: A \rightarrow B$ is continuous and 2-to-1, and A is compact. Then there is an increasing sequence $W_1, W_2, \dots, W_\beta, \dots, W_\alpha$ of open sets in A such that (1) $W_\alpha = A$, (2) each $W_\beta = (W_\beta)^\wedge$, (3) each $W_\beta \setminus \bigcup\{W_j : j < \beta\}$ is the union of two nonempty separated twin sets V_β and $(V_\beta)^\wedge$, and (4) if Z' is $f^{-1}(Z)$ for some compactum Z in the image B , and β is the least ordinal such that Z' is contained in W_β , then Z' intersects $V_\beta \cup (V_\beta)^\wedge$, and the twin function maps $Z' \cap V_\beta$ homeomorphically onto $Z' \cap (V_\beta)^\wedge$.

Example. In the drawing above, the arrows indicate the identifications which make up the 2-to-1 map from the dendrite D onto the circle S .

Note that the point a is a hinge point since it is a limit of two twin sequences, one from the vertical segment at a , the other from the horizontal segment $[a, b]$. Note also that the twin map is not continuous at a since one of the sequences just described, $\{x(i)\}$, converges to a but the twin sequence $\{x(i)^\wedge\}$ does not converge to the twin of a , namely b . The interior lemma says for instance that since the segment $[a, b]$ maps onto $f(D)$, then the segment $[a, b]$ must have interior in D . As regards Reference Theorem 2, W_1 might be all points of D of order two, V_1 might be the points of order two on vertical segments and so V_1^\wedge would then be the points of order two on $[a, b]$. Next W_2 might be $(D - W_1) - \{a, b\}$, V_2 might be the points of W_2 of order one, and then V_2^\wedge would be the points of W_2 of order three. Finally, the last W , W_3 , would be $\{a, b\}$.

III. 2-TO-1 MAPS ON HEREDITARILY INDECOMPOSABLE CONTINUA

Suppose f is a 2-to-1 continuous function from the hereditarily indecomposable continuum X onto a continuum Y . There is a subcontinuum of Y which is minimal with respect to its inverse under f being connected. No inverse is ever degenerate for a 2-to-1 map, so the restriction of f still maps a hereditarily indecomposable continuum 2-to-1 onto a continuum. We assume however that f has the stronger property (Assumption 1) that each connected nondense set in Y has disconnected preimage. This is less justified than the minimality assumption but we show in Theorem 1 that Assumption 1 dictates a clear structure for f . The proof of Theorem 1 relies heavily on the structure of 2-to-1 maps summarized in Reference Theorem 2.

Theorem 1. Suppose f maps the hereditarily indecomposable metric continuum X (exactly) 2-to-1 onto the continuum Y . Then the following are equivalent:

- (1) f satisfies Assumption 1, i.e. if Y' is a connected nondense subset of Y then Y' has a disconnected preimage,
- (2) if Z is a proper subcontinuum of Y , then the preimage of Z consists of two disjoint continua and f maps each homeomorphically onto Z , and
- (3) Y is hereditarily indecomposable and no proper subcontinuum of Y has connected preimage.

Proof (2) implies (3). If C is a proper subcontinuum of Y then property (2) immediately implies that C is decomposable. If Y itself is decomposable, $Y = A \cup B$, two proper subcontinua, then each of A and B have preimages prescribed by property (2). Hence $f^{-1}(A) = A1 \cup A1^\wedge$ and $f^{-1}(B) = B1 \cup B1^\wedge$, where one of the A 's must intersect one of the B 's, say $A1$ and $B1$ intersect. But then $A1$ is a subset of $B1$ (or the other way about) so $A1^\wedge$ is a subset of $B1^\wedge$; this means the continuum X (the domain of f) is the disconnected union of $B1$ and $B1^\wedge$. \square

Proof (3) implies (1). This follows immediately from the fact that f is 2-to-1 and Cook's result in [1] that f must be confluent. \square

Proof (1) implies (2). The proof is inductive. Suppose $W1, W2, \dots, W_\alpha$ is a sequence of subsets of X as described in Reference Theorem 2.

First, for the base case, suppose Z is a proper subcontinuum of Y and $Z' = f^{-1}(Z)$ is contained in the first W , $W1 = V1 \cup V1^\wedge$. The twin disjoint open sets $V1$ and $V1^\wedge$ each intersect Z' in a compactum. Since f is 1-to-1 on any subset of $V1$ (or of $V1^\wedge$), each of $Z' \cap V1$ and $Z' \cap V1^\wedge$ is a continuum since each is homeomorphic to Z . Hence property (2) is true in this case.

Now assume that β is an ordinal with the inductive property that if M is a proper subcontinuum in Y whose inverse M' is a subset of W_j for some $j < \beta$, then $M' = C \cup C^\wedge$, two disjoint twin continua. Let Z be a proper subcontinuum of Y whose inverse Z' is contained in W_β . Let $S_\beta = \bigcup \{W_j : j < \beta\}$; then, as given in Reference Theorem 2, $W_\beta = S_\beta \cup V_\beta \cup (V_\beta)^\wedge$, where V_β and $(V_\beta)^\wedge$ are separated twin sets and S_β is open. We may assume that Z' is not contained in S_β since otherwise Z' is contained in some previous W_β (Reference Theorem 2). Let N denote the set of components of $Z' \cap S_\beta$. We will show:

- (i) each member of N is separated from its twin, and
- (ii) if each member of N is separated from its twin, then $Z' = C \cup C^\wedge$, two disjoint twin continua.

Fact 1. (Proof later in paper.) If the connected subset d of Z' contains no hinge point for points of $d \cup d^\wedge$, then d and d^\wedge are separated connected sets.

Fact 2. (Proof later in paper.) Suppose T is a continuum in Z' such that for each x in $T \cap S_\beta$ there is a connected set C containing x and contained in

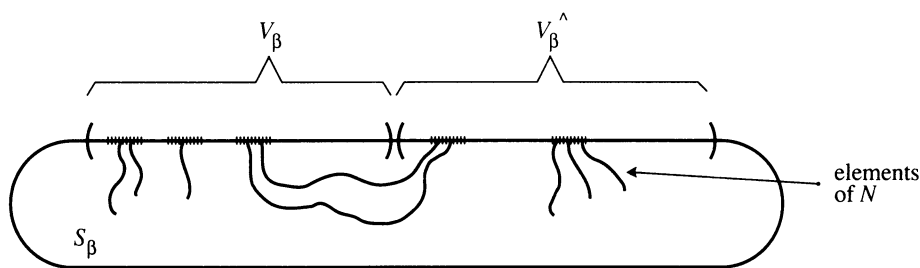


ILLUSTRATION 2

$T \cap S_\beta$ such that (a) C and C^\wedge are separated, and (b) some point of $W_\beta \setminus S_\beta$ is a limit point of C . Then T and T^\wedge are separated.

Fact 3. (Proof later in paper.) If k is in N , then some point of $W_\beta \setminus S_\beta$ is a limit point of k .

Proof of (i). Illustration 2 is a limited drawing of N although, of course, N certainly contains no arcs and is probably infinite.

First note that if M is a compactum in Z whose inverse M' is in S_β then M' is in some W_j for $j < \beta$ (Reference Theorem 2). Hence the induction hypothesis holds for any such M .

Now let k denote a member of N . We wish to show that k is separated from its twin by using Fact 2 with $T = \text{Cl}(k)$, the closure of k . By Fact 3, T intersects $W_\beta \setminus S_\beta$. Now let x be a point of $k = T \cap S_\beta$. Let $C(1), C(2), \dots$, denote an increasing sequence of continua in k such that x belongs to $C(1)$ and the union C of all of the $C(i)$ has a limit point in $W_\beta \setminus S_\beta$. One such sequence might be constructed by taking components (containing x) of T minus smaller and smaller neighborhoods of $T \cap (W_\beta \setminus S_\beta)$. By the induction hypothesis, each $C(i)$ is separated from its twin $C(i)^\wedge$, a continuum (use Fact 1 to get it connected) containing x^\wedge . If C is not separated from its twin C^\wedge (equal to the union of the $C(i)^\wedge$) then $C \cup C^\wedge$ is a connected set equal to its twin, contrary to Assumption 1. Thus all of the conditions of Fact 2 are met, so $\text{Cl}(k)$, and hence k , is separated from its twin. \square

Proof of (ii). We first prove that every continuum T in Z' that intersects $W_\beta \setminus S_\beta$ is separated from its twin. To see this, note that each component of $Z' \cap S_\beta$ is separated from its twin from part (i), so each component C of $T \cap S_\beta$ must also be separated from its twin. Then, since T is connected, C must have a limit point in $W_\beta \setminus S_\beta$ from the “to the boundary” theorem, in [5] for instance. Thus for each x in $T \cap S_\beta$ the component of $T \cap S_\beta$ containing x satisfies the hypothesis of Fact 2.

Second we prove that if k is in N then k^\wedge is in N . From part (i) we know that k is separated from its twin, so its twin k^\wedge is connected by Fact 1. But

if k^\wedge is properly contained in some k' of N then k is properly contained in the connected set $(k')^\wedge$ of $Z' \cap S_\beta$ (a contradiction); hence k^\wedge is in N also.

To complete the proof of part (ii) we will (painfully) divide Z' into finitely many "strings", each consisting, loosely, of chains of elements of N plus various points from $W_\beta \setminus S_\beta$. It will follow from the construction that the strings are disjoint, and we will show that each string is compact and that the twin of any string is another string. Thus the strings can be divided into two groups R_1, R_2, \dots, R_j and $R_1^\wedge, R_2^\wedge, \dots, R_j^\wedge$. But now f is 1-to-1 on the compactum $R = R_1 \cup R_2 \cup \dots \cup R_j$ so that R must be a continuum since it is homeomorphic to Z' (and j must be 1). Thus Z' is the union of the two disjoint twin continua R and R^\wedge , as required to complete the proof of part (ii).

Let ε' denote one fourth the distance between the disjoint twin compacta $A = Z' \cap V_\beta$ and $B = Z' \cap (V_\beta)^\wedge$. For each positive number $\varepsilon < \varepsilon'$ let $G(\varepsilon)$ denote a finite open (in X) cover of A using sets of diameter less than ε that intersect A . This cover generates a decomposition of A into finitely many disjoint compacta, $A = D_1' \cup \dots \cup D_m'$ where each D_j' is the intersection with A of the union of a maximal coherent subcollection of $G(\varepsilon)$. (See Illustration 3.)

Now let E denote the collection $\{D_i' : i = 1, \dots, m\} \cup \{(D_i')^\wedge : i = 1, \dots, m\}$. We say that the elements D_i and D_j of E are *equivalent* if D_i and D_j are the first and last terms of a finite sequence D_1, D_2, \dots, D_n from E for which there exist elements $k(i)$, $i = 1, \dots, n-1$, of N such that for each i , $k(i)$ has a limit point in D_i and in $D(i+1)$. The D 's and k 's form a "thread" from D_i to D_j , but their union is probably not connected since the D 's need not be connected. The set E and the equivalence relation change with ε .

We wish to establish that there is an $\varepsilon < \varepsilon'$ such that no D generated by the cover $G(\varepsilon)$ is equivalent to its twin D^\wedge . On the contrary, suppose there is a sequence $\{\varepsilon(i)\}$ of positive numbers with zero limit such that for each i some $D(i)$ and $D(i)^\wedge$ in $E(i)$ are equivalent. For each i , let $L(i)$ denote a thread from $D(i)$ to $D(i)^\wedge$, more precisely, the union of the elements of $E(i)$ and the closures of the members of N that form a thread from $D(i)$ to $D(i)^\wedge$. (See Illustration 4.)

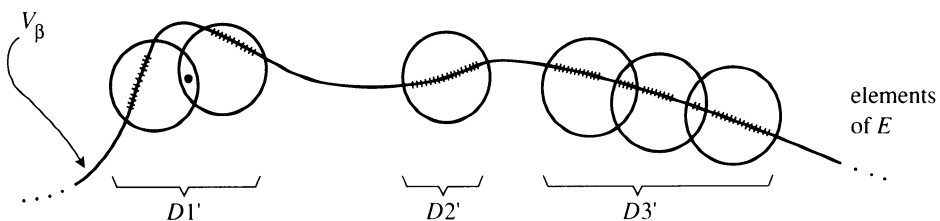


ILLUSTRATION 3

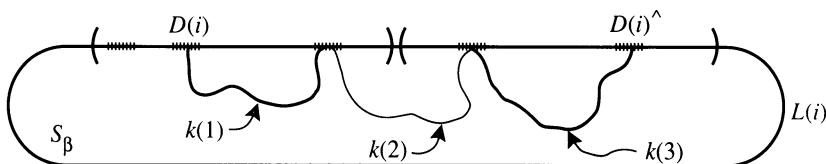


ILLUSTRATION 4

Let L denote the limit of a convergent subsequence of the compacta $\{L(i)\}$. Note that L is compact. We will show in the next paragraph that L is connected. Thus L is a continuum in Z' that intersects $W_\beta \setminus S_\beta$ and, as was established earlier, L is separated from its twin. But each $L(i)$ contains a pair of twin points $x(i)$ and $x(i)^\wedge$ from $D(i)$ and $D(i)^\wedge$ respectively, and the limits of the two sequences $\{x(i)\}$ and $\{x(i)^\wedge\}$ must be a twin pair x and x^\wedge in L since there is no point common to the two compacta $Z' \cap V_\beta$ and $Z' \cap (V_\beta)^\wedge$. The existence of x and x^\wedge in L contradicts the fact that L is separated from its twin. This contradiction establishes the existence of the ε with the required properties.

Now suppose that L is not connected and L is contained in the union of, and intersects each of, two open (in X) sets R and S whose closures are disjoint. Each $L(i)$ is contained in $R \cup S$ for large i and must intersect both. Note that, since $A = Z' \cap V_\beta$ is homeomorphic to $B = Z' \cap (V_\beta)^\wedge$ via the twin function (Reference Theorem 2), if, for each i , $g(i1)$ and $g(i2)$ are intersecting elements of the $\varepsilon(i)$ cover of A , then the limit (as $i \rightarrow \infty$) of the diameters of the sets $(g(i1)^\wedge \cup g(i2)^\wedge) \cap B$ is zero, and so, for large i , the diameters of these sets and the sets $(g(i1) \cup g(i2)) \cap A$ may be assumed to be less than $1/4$ the distance between the closures of R and S . Thus for large i , if D is in $E(i)$ and is contained in $L(i)$ then D lies wholly in R or in S , even though D may not be connected. (This follows from the fact that D or D^\wedge is the intersection with A of a union of a coherent collection of small open sets each element of which intersects A .) Thus each D part of $L(i)$ cannot intersect both R and S , nor can the closure of any member of N in $L(i)$ since each is connected, so $L(i)$ lies wholly in R or in S for large i . This contradicts L being their limit and intersecting both R and S . Hence L must be connected.

So, finally, we are in a position to define the finitely many "strings" whose union is Z' . Suppose ε is a positive number such that no D is equivalent to its twin. For each k in N , let $D(k)$ denote the elements of E (for this ε) which contain a limit point of k . Each $D(k)$ is nonempty by Fact 3 and, of course, each two elements of $D(k)$ are equivalent. Now that we have a suitable ε and E , we will introduce a little color into the argument, both for its helpful mental image and for concise descriptions. To begin with, paint each element of E so that D and D' are equivalent iff they are the same color. Then, for each k in N , the members of $D(k)$ are all the same color; paint k this same color. We have now colored each point of Z' using finitely many colors (since

E is finite). For each color used, the set of points of that color is a "string". As mentioned before, to complete part (ii) we need to show that (1) the twin of any string is a string, and (2) each string is compact.

(1) We wish to show that the twin of any string is a string. Suppose k is in N . Since X is hereditarily indecomposable and k and k^\wedge are separated members of N (established earlier), D belongs to $D(k)$ iff D^\wedge belongs to $D(k^\wedge)$. This means that if L is a thread of, say, blue points, then L^\wedge is red (or some other color besides blue). Thus it follows from the definition of the equivalence class that the twin of the blue string is the red string.

(2) We now wish to show that each string is compact. Suppose again that blue is one of the colors used, and let C denote the string of blue points. There are only finitely many blue elements of E and each is compact, but there may be infinitely many blue k 's from N in C . Suppose C is not compact; there is then a sequence $k(i)$ of blue elements of N that converges to a continuum T containing, say, a red point p . Since there are only finitely many D 's in E we may assume that $D(k(i)) = D(k(j))$ for each $k(i)$ and $k(j)$ from the blue sequence. We first wish to establish that T intersects both a red member of E and a blue member of E . Suppose D is in $D(k(1))$; since each $k(i)$ has the same $D(k(i))$, D has a limit point of each $k(i)$, so some (blue) point of D is in T . If the red point p of T belongs to some red k' of N then each D in $D(k')$ is red. Since $\text{Cl}(k')$ and T are intersecting continua and X is hereditarily indecomposable, $\text{Cl}(k')$ is contained in T (since T has two colors and $\text{Cl}(k')$ has only one color). Hence some point p' from some red D in $D(k')$ is in T , and T intersects at least two members of E , one red and one blue.

We will now show that no continuum in Z' can intersect two members of E of different colors. This contradiction to the existence of T established in the previous paragraph will prove that C is compact as wanted. So assume that M is a continuum in Z' minimal with respect to intersecting two members of E of different colors. (A minimal such M exists with the help of Zorn's lemma and the fact that there are only finitely many D 's of any color, each compact, and there are only finitely many colors.) Divide M into, say, its brown points, B , and its nonbrown points, R . Since M is connected, B and R cannot be separated so without loss of generality we assume there is a sequence $\{b(i)\}$ of brown points in B with nonbrown, say red, sequential limit point b in R . Since the D 's are finite in number and compact, we will assume each $b(i)$ is in S_β ; and hence each $b(i)$ is in some brown member $k(i)$ of N . We know each $\text{Cl}(k(i))$, and hence $k(i)$, is in M since, as before, $\text{Cl}(k(i))$ and M are intersecting continua, M has at least two colors and $\text{Cl}(k(i))$ has only one color. Now assume that $\{k(i)\}$ converges to a continuum M' in M . Exactly as before with T , since M' must contain the red point b , M' intersects a red and a brown member of E , so $M' = M$. Then M is the limit of the $k(i)$ sequence and each $k(i)$ is a connected subset of $M \cap S_\beta$. Hence $M \cap S_\beta$ is

connected and so $M \cap S_\beta$ is a subset of a single brown k of N . But since M is the limit of the $k(i)$ sequence, M is contained in the closure of $M \cap S_\beta$ and so is contained in $\text{Cl}(k)$. This means that all of the D that intersect M must also be brown. Thus it is not possible for any continuum in Z' to intersect inequivalent D 's from E . \square

This completes the proof of (1) implies (2) except for Facts 1, 2, and 3:

Proof of Fact 1. If the connected subset d of Z' contains no hinge point for points of $d \cup d^\wedge$, then d and d^\wedge are separated connected sets.

If d^\wedge is connected and not separated from d then $d \cup d^\wedge$ is a connected set that violates Assumption 1, part of the hypothesis of the theorem. Hence we suppose that $d^\wedge = A \cup B$, two separated, nonempty sets. Since the connected set d is $A^\wedge \cup B^\wedge$, and A^\wedge and B^\wedge are disjoint, some p in A^\wedge (say), and hence in d , is a limit of a sequence of points $\{p(i)\}$ from B^\wedge . Since each $p(i)^\wedge$ is in B , the limit of the sequence $\{p(i)^\wedge\}$ is in B , and either p or p^\wedge is this limit due to the semicontinuity of the twin function. Since p^\wedge is in A , p must be the limit of both sequences $\{p(i)\}$ and $\{p(i)^\wedge\}$. This means that p is a hinge point for points of $d \cup d^\wedge$ in contradiction to the hypothesis. Hence d^\wedge is connected and separated from d . \square

Proof of Fact 2. Suppose T is a continuum in Z' such that for each x in $T \cap S_\beta$ there is a connected set C containing x and contained in $T \cap S_\beta$ such that (a) C and C^\wedge are separated, and (b) some point of $W_\beta \setminus S_\beta$ is a limit point of C . Then T and T^\wedge are separated.

If T contains no pair of twin points $\{x, x^\wedge\}$ from $W_\beta \setminus S_\beta$ then set $S = T$, otherwise let S be a subcontinuum of T minimal with respect to containing some twin pair of points from $W_\beta \setminus S_\beta$. Assume that T and T^\wedge are not separated. We will analyze the structure of S and S^\wedge and find a contradiction.

Note that, whether S is T or not, S and S^\wedge are not separated since no continuum containing twin pairs of points is even disjoint with its twin. It follows from Fact 1 that S has a hinge point p for points of $S \cup S^\wedge$. Let $\{p(i)\}$ in S and $\{p(i)^\wedge\}$ in S^\wedge denote sequences that converge to p . T is in W_β so the sequences must be subsets of S_β since twin sequences in $T \cap (W_\beta \setminus S_\beta)$ have distinct limits. By the Fact 2 hypothesis there is, for each i , a connected set $C(i)$ in $T \cap S_\beta$ containing $p(i)$ such that (a) $C(i)$ is separated from its twin $C(i)^\wedge$ and (b) some point $z(i)$ of $W_\beta \setminus S_\beta$ is a limit point of $C(i)$. By Fact 1, $C(i)^\wedge$ must also be connected. Since the closures of $C(i)$ and $C(i)^\wedge$ must be disjoint (neither is contained in the other and X is hereditarily indecomposable), $z(i)^\wedge$ must be in the closure of $C(i)^\wedge$, thanks to the semicontinuity of the twin function. Note that the closure of $C(i)$ is in S rather than the other way around since S is not separated from its twin. By taking subsequences we assume $z = \lim\{z(i)\}$ is in $K = \lim\{C(i)\}$ and $z^\wedge = \lim\{z(i)^\wedge\}$ is in $L = \lim\{C(i)^\wedge\}$. (See Illustration 5.)

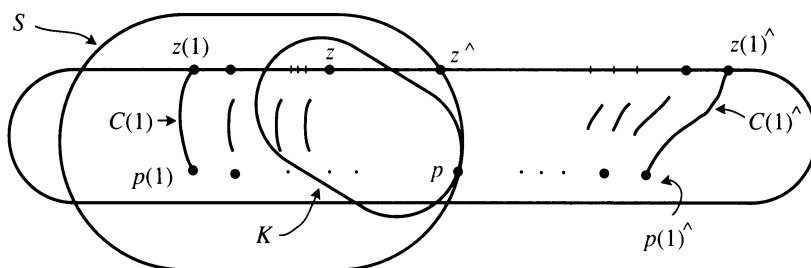


ILLUSTRATION 5

Because p belongs to K , L , and S , the three continua are nested. Each $C(i)$ contained in S means K is in S . Each $C(i)^\wedge$ will miss S or be contained in S so there are two cases; (Case 1) infinitely many $C(i)^\wedge$ miss S , and (Case 2) infinitely many $C(i)^\wedge$ are contained in S .

In Case 1 the Reference Interior Lemma will be used. First we want to show that L is a proper subset of S . Since f is exactly 2-to-1 on the compactum $S \cup S^\wedge$ and since the compactum S maps onto the image $f(S \cup S^\wedge)$, S must have interior relative to $S \cup S^\wedge$. Hence, since infinitely many $C(i)^\wedge$ are in $S^\wedge \setminus S$, the limit L of the $C(i)^\wedge$ cannot contain S . Hence L is a proper subset of S .

The larger of K and L contains both z and z^\wedge from $W_\beta \setminus S_\beta$ and since both K and L are in S , S contains both z and z^\wedge . This means that S must be minimal with respect to containing such a twin pair from $W_\beta \setminus S_\beta$ by the definition of S . Hence S is equal to the larger of K and L , which must be K since L is a proper subset of S .

Let x be in $K \setminus L$. Now x in K means there is a sequence of $x(i)$ from $C(i)$ converging to x . But each $x(i)$ is in $C(i)$, a subset of S . Since $S = K$ and K is the limit of the sequence $\{C(i)\}$, there is, for each i , a sequence $\{y(i, j)\}_j$ converging to $x(i)$, with each $y(i, j)$ in some $C(j')$. Its twin sequence $\{y(i, j)^\wedge\}$ converges to either $x(i)$ or $x(i)^\wedge$. But $x(i)^\wedge$ is in $C(i)^\wedge$ which is outside of S in Case 1, and any limit of $C(j')$ points is in K by definition of K , so $\{y(i, j)^\wedge\}$ converges to $x(i)$. This means each $x(i)$ and hence x is in L , a contradiction.

In Case 2, we assume that each $C(i)^\wedge$ as well as $C(i)$ is in S . This means that the limit of the $C(i)^\wedge$, L , is in S and, as before, S is equal to the larger of K and L . If $S = K$ then S is the limit of the $C(i)$, so $C = \bigcup \{C(i) \cup C(i)^\wedge : i = 1, 2, \dots\}$ contradicts Assumption 1 since C is connected and $C = C^\wedge$. If $S = L$ then S is the limit of the $C(i)^\wedge$, and exactly the same contradiction is reached.

Either way there is a contradiction, so Fact 2 is proved. \square

Proof of Fact 3. If k is in N then some point of $W_\beta \setminus S_\beta$ is a limit point of k .

From Reference Theorem 2, each $Wj = Wj^\wedge$ so $S_\beta = (S_\beta)^\wedge$, and since

S_β is open, $f(S_\beta) = S$ is open in Y . Since k is a connected subset of S_β , $\text{Cl}(f(k))$ is a continuum. If $\text{Cl}(f(k))$ contains the boundary point y of S , then one point of $f^{-1}(y)$ is a point of $W_\beta \setminus S_\beta$ that is a limit point of k . Hence we assume that $\text{Cl}(f(k))$ is contained in S . Since the continuum Z is not in S , the component of $Z \cap S$ that contains $\text{Cl}(f(k))$ goes to the boundary of S . Hence, there is a continuum K in $Z \cap S$ containing $\text{Cl}(f(k))$ properly. By the induction hypothesis, the inverse of K is $C \cup C^\wedge$ two disjoint twin continua in S_β . But k must be a proper subset of either C or C^\wedge , contradicting the fact that k is a component of $Z' \cap S_\beta$. \square

GLOSSARY

1. A continuum X is *arclike* if for each positive number ε , there is an ε -map (each point inverse has diameter less than ε) from X onto the interval $[0, 1]$.
2. A collection G of sets is *coherent* if for any two elements g and g' of G there is a finite sequence $g_1, g_2, g_3, \dots, g_n$ of elements of G such that $g = g_1$, $g' = g_n$, and for each relevant i , g_i and g_{i+1} intersect in a nonempty set.
3. A map f is *confluent* if for each continuum C in the image, each component of the inverse of C maps onto C .
4. A space is a *continuum* if it is both connected and compact. (In this paper it is also assumed to be metric.)
5. A function is *finitely discontinuous* if it has at most finitely many discontinuities.
6. A continuum is *hereditarily indecomposable* if each of its subcontinua is indecomposable.
7. A continuum is *indecomposable* if it is not the sum of two of its proper subcontinua.
8. Two sets are *separated* if they are disjoint and neither has a limit point of the other.
9. A continuum X is *tree-like* if for each positive number ε there is an ε -map from X onto a tree, i.e. a connected acyclic finite graph.

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